J. DIFFERENTIAL GEOMETRY
39 (1994) 35-50

GLOBAL EXISTENCE AND CONVERGENCE OF YAMABE FLOW

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1. Introduction

Let M^n be a closed connected manifold of dimension $n \geq 3$ and $[g_0]$ a given conformal class of metrics on M. We consider the (normalized) total scalar curvature functional $\mathbf{S}$ on $[g_0]$,

$$\mathbf{S}(g) = \frac{1}{V(g)^{(n-2)/n}} \int_M R_g \, dv_g, \qquad g \in [g_0],$$

where dv_g is the volume form of g, $V(g) = \int_M dv_g$, and R_g denotes the scalar curvature function of g. Simple computations [1], [13] show that the gradient of $\mathbf{S}$ at g is given by $((n-2)/2n)V(g)^{-1}(R_g - s_g)g$, where $s_g = V(g)^{-1} \int_M R_g \, dv_g$. The negative gradient flow of $\mathbf{S}$ is hence given by

$$\frac{\partial g}{\partial t} = \frac{n-2}{2n} V(g)^{-1}(s_g - R_g)g. \tag{1.1}$$

This flow preserves the volume, as can be easily verified. Changing time by a constant scale, we then arrive at the *Yamabe flow* introduced by Hamilton:

$$\frac{\partial g}{\partial t} = (s - R)g. \tag{1.2}$$

(The subscript g is omitted.) Along Yamabe flow, the total scalar curvature is decreased. Moreover, if the flow exists for all time and converges smoothly as $t \to \infty$, then the limit metrics have constant scalar curvature. Hence, Yamabe flow should be an efficient tool to produce metrics of constant scalar curvature in a given conformal class. Indeed, it was originally conceived to attack the Yamabe problem. Then the Yamabe

Received November 4, 1991. The author was partially supported by NSF.

problem was resolved by Schoen [13] using a different approach. The significance of Yamabe flow is that it is a natural geometric deformation to metrics of constant scalar curvature. Moreover, this flow provides a natural approach to Morse theory for the total scalar curvature functional. One also notes that Yamabe flow corresponds to the fast diffusion case of the porous medium equation (the plasma equation) in mathematical physics (see (1.4)). There is an extensive literature on local analysis of the porous medium equation.

In order to state the results conveniently, we introduce some terminology.

Definition. We say that $[g_0]$ is *scalar positive, scalar negative,* or *scalar flat,* if $[g_0]$ contains a metric of positive, negative, or identically zero scalar curvature respectively. (It is well known that these three cases are mutually exclusive and exhaust all possibilities.)

We first consider the scalar positive case.

Theorem 1. *Assume that* $[g_0]$ *is scalar positive. Assume in addition that* $(M, [g_0])$ *is locally conformally flat. Then for any given initial metric in* $[g_0]$, *the flow* (1.2) *has a unique smooth solution on the time interval* $[0, \infty)$. *Moreover, the solution metric* g *converges smoothly to a unique limit metric of constant scalar curvature as* $t \to \infty$.

We do not know whether the convergence rate is exponential. But such is the case, at least when the limit metric is a stable critical point of the total scalar curvature functional. Our methods do not extend directly to manifolds which are not locally conformally flat. We hope to treat them in a forthcoming paper.

In general, a scalar positive conformal class can contain constant scalar curvature metrics which are saddle points of the total scalar curvature functional [15]. Starting near a saddle point and in a descending direction, a negative gradient flow will go away from that point. It is a rare phenomenon that a negative gradient flow (or gradient flow) always converges in the presence of saddle points, unless the flow is defined on a compact space or is a linear system of equations. Indeed, at least in a geometric context, Theorem 1 seems to be the first example of this phenomenon in an infinite dimensional space, where the flow equation is a nonlinear parabolic equation depending on more than one non-time variable. (The gradient flow of the classical energy functional for curves has one non-time variable; the gradient flow of the symplectic action as studied by Floer is a Cauchy-Riemann equation.)

Another remark is that the round sphere S^n is included in Theorem 1. Since it has been a tradition to emphasize the difference between S^n and other manifolds in the context of the Yamabe problem, it might appear unexpected that the Yamabe flow on S^n always converges.

Next we consider the scalar negative and flat cases.

Theorem 2. *Assume that* $[g_0]$ *is either scalar negative or scalar flat. Then for any given initial metric in* $[g_0]$, *the solution of* (1.2) *exists for all time and converges smoothly to a unique limit metric of constant scalar curvature at exponential rate as* $t \to \infty$.

We also present a proof of the general long time existence theorem.

Theorem 3. *For any given initial metric, the flow* (1.2) *has a unique smooth solution on the time interval* $[0, \infty)$.

Previous results on Yamabe flow are as follows. Long time existence was first obtained by Hamilton (see [8]), but his proof has not been published. Our argument is different from his. In case the initial metric has negative scalar curvature, Hamilton has shown the convergence of the flow [8], and also that the scalar curvature converges to a constant along Yamabe flow, provided that the initial metric has positive scalar curvature. But the resulting convergence is not strong enough to imply the convergence of the flow itself (see [8]). For the special case that the initial metric has positive Ricci curvature and is locally conformally flat, Chow [3] has been able to extend the arguments in [7] to obtain the convergence. On the other hand, the Ricci flow on surfaces—the 2-dimensional analogue of Yamabe flow—has been completely solved by Hamilton [8] and Chow [2] (it always converges).

Now we discuss the proofs of the above results. The proof of Theorem 1 is based on a Harnack inequality. Fix a background metric $g_0 \in [g_0]$ and write $g = u^{4/(n-2)} g_0$ with u denoting a positive function. Then (1.2) can be written in the equivalent form

$$(1.3) \qquad \frac{\partial u^N}{\partial t} = (n-1)N(L_{g_0} u + c(n) s u^N), \quad \text{with } N = \frac{n+2}{n-2} \quad \text{and} \quad c(n) = \frac{n-2}{4(n-1)},$$

or, in consequence of changing time by a constant scale,

$$(1.4) \qquad \frac{\partial u^N}{\partial t} = L_{g_0} u + c(n) s u^N,$$

where L_{g_0} is the conformal Laplacian of g_0:

$$L_{g_0} u = \Delta_{g_0} u - c(n) R_{g_0} u.$$

We shall identify (1.4) with (1.3). Notice that in terms of u, the volume form dv of g is $u^{2n/(n-2)}\,dv_{g_0}$, the scalar curvature of g is

$$R = -\frac{1}{c(n)}\,\frac{L_{g_0}u}{u^N}, \tag{1.5}$$

and hence the average scalar curvature s is

$$s = \frac{1}{c(n)\int_M u^{2n/(n-2)}\,dv_{g_0}} E(u), \tag{1.6}$$

where

$$E(u) = \int_M (|\nabla_{g_0} u|^2 + c(n) R_{g_0} u^2)\,dv_{g_0}$$

is the energy. Thus we obtain the following Harnack inequality for the solution u:

$$\frac{|\nabla_{g_0} u|}{u} \leq C \quad \text{or} \quad \inf_t u \geq c \sup_t u. \tag{1.7}$$

With this inequality, the long time existence and subconvergence at ∞ follow easily. The convergence to a unique limit then follows from Simon's unique asymptotical limit theorem [18]. The Harnack inequality (1.7) is interesting for its own sake. Indeed, the general local (weak) Harnack inequality does not seem to hold for equations of the type (1.4) because the exponent N on the left-hand side is too big. In some sense the critical exponent is $n/(n-2)$; namely, a certain "intrinsic" Harnack inequality holds if N is replaced by any positive number $< n/(n-2)$ (see [5]). For comparison, one notes that the critical exponent for the maximum principle of Moser type is N. One would expect that the maximum principle of Moser type holds under a small integral assumption at the critical exponent N. But this is still an open problem. In any case, local analysis of (1.4) is rather delicate.

The proof of the Harnack inequality (1.7) depends on two global arguments. The first is the injectivity of the developing map from the universal cover of $(M, [g_0])$ into S^n. The second is the Alexandrov reflection principle. These two arguments were used by Schoen in [15] for proving the compactness of constant scalar curvature metrics on locally conformally flat manifolds which are not covered by the round sphere. The Alexandrov reflection principle was used earlier by Serrin [17] and Gida, Ni, and Nirenberg [6] to obtain the symmetry properties of positive solutions of certain nonlinear PDEs.

The proof of Theorem 2 is fairly elementary and essentially uses only the maximum principle. We prove Theorem 3 in the following manner.

Since the solution u can at most grow at an exponential rate by the maximum principle, the long time existence will follow if we can show that u can never approach zero in finite time. We observe that the constancy of volume together with an estimate of modulus of continuity for solutions of the plasma equation (due to DiBenedetto [4] and Sacks [12]) implies that u can never become identically zero in finite time. A comparison argument based on the maximum principle then demonstrates that u remains positive always.

Applications of Theorem 1 to Morse theory for the total scalar curvature functional will be discussed in a forthcoming paper.

We are grateful to Professors R. Schoen, R. Hamilton, and M. Struwe for many helpful and stimulating discussions on the subject. In particular, we thank Prof. Schoen for bringing our attention to his paper [15]. We also acknowledge discussions with Professors M. Crandall and E. DiBenedetto on relevant literature on the porous medium equation.

2. The scalar positive case

We start with the short time existence.

Proposition 1. *For each $\delta > 0$ and $\Lambda > 0$ there is some $T > 0$ depending on δ, Λ, n and the background metric g_0 with the following properties. If u_0 is a positive smooth function on M such that $u_0 \geq \delta$, $\| u_0 \|_{C^3} \leq \Lambda$ (norm measured in g_0), then* (1.4) *with initial data u_0 has a unique positive smooth solution on the time interval* $[0, T]$.

This follows from the linear theory and the implicit function theorem in a standard way. Although not necessary, it is possible to avoid dealing directly with integration involved in s by rescaling in time. Namely, one first solves the equations

$$\frac{\partial g}{\partial t} = -Rg \quad \text{or} \quad \frac{\partial u^N}{\partial t} = Lu$$

and then rescales in time to arrive at (1.2) or (1.4).

Now we assume that $(M, [g_0])$ is locally conformally flat and that $[g_0]$ is scalar positive. Choose a background metric g_0. Let an initial metric g^0 in $[g_0]$ be given. By Proposition 1, we have a unique smooth solution g for (1.2) with $g(0) = g^0$ on a maximal time interval $[0, T^*)$, and the corresponding solution u of (1.4) will be denoted by u.

We first consider the case that $(M, [g_0])$ is not conformally covered by S^n. By a theorem of Schoen and Yau [16], there is a conformal diffeomorphism Φ from the universal cover $\widetilde{M}$ of $(M, [g_0])$ onto a dense domain Ω of S^n. Thus $(M, [g_0])$ is the quotient of Ω under a Kleinian group, and $\Gamma = \partial\Omega$ is the limit set of this group. Note $\Gamma \neq \varnothing$. We set $\tilde{g} = (\Phi^{-1})^* \pi^* g$ and $\tilde{g} = \tilde{u}^{4/(n-2)} g_{S^n}$, where $\pi: \widetilde{M} \to M$ is the covering map, and g_{S^n} denotes the round sphere metric. Then $\tilde{g}$ solves (1.2) and $\tilde{u}$ solves (1.4) with g_0 replaced by g_{S^n}. The following lemma is a corollary of Proposition 2.6 in [16].

Lemma 1. *For each fixed $T \in [0, T^*)$, $\tilde{u}(p, t)$ approaches ∞ uniformly for all $t \in [0, T]$ as p approaches the limit set Γ.*

Given $p_0 \in M$, we choose a point $\bar{p}_0 \in \widetilde{M}$ and a neighborhood V of $\bar{p}_0$ such that $\pi(\bar{p}_0) = p_0$ and $\operatorname{dist}(\Phi(V), \Gamma) > 0$. Then there is some $C > 0$ such that

$$\tilde{u}_0 > C^{-1}, \quad \| \tilde{u}_0 \|_{C^4} \leq C \quad \text{on } \Phi(V), \tag{2.1}$$

where $\tilde{u}_0 = \tilde{u}(\cdot, 0)$, and the norm is measured in g_{S^n}. By Lemma 1, we can actually assume that $\tilde{u}_0 > C^{-1}$ holds everywhere.

For a fixed $q_0 \in \Phi(V)$, let $F: S^n \to \mathbb{R}^n$ be the stereographic projection with q_0 as north pole. To be more specific, F denotes the inverse of the map

$$F^{-1}(x) = \left(\frac{2x}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1} \right), \qquad x \in \mathbb{R}^n,$$

and we set $q_0 = (0, 0, \cdots, 1)$. We also introduce the following coordinates around q_0:

$$G(x) = \left(\frac{2x}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right), \qquad |x| \leq 1.$$

Note that $G(0) = q_0$ and that G is the composition of F^{-1} with the inversion $x/|x|^2$. A simple computation leads to

Lemma 2. *Let f be a positive smooth function on $\Phi(V)$. Set $(F^{-1})^* \cdot (f^{4/(n-2)} g_{S^n}) = \bar{f}^{4/(n-2)} g_{\mathbb{R}^n}$, where $g_{\mathbb{R}^n}$ denotes the standard metric of $\mathbb{R}^n$. Define $a_0 = f(q_0)$, $a_i = \partial(f \circ G)(o)/\partial x^i$, and $a_{ij} = \partial^2 (f \circ G)(o)/\partial x^i \partial x^j$. Then $\bar{f}$ has the following asymptotic expansion near ∞ (using summation* *convention*):

$$\tag{2.2}\begin{aligned}\bar f &= \frac{2^{(n-2)/2}}{|x|^{n-2}}\\ &\quad\cdot\left(a_0+\frac{a_ix_i}{|x|^2}+(a_{ij}-\frac{n-2}{2}a_0\delta_{ij})\frac{x_ix_j}{|x|^4}+O\left(\frac{1}{|x|^3}\right)\right),\\ \frac{\partial\bar f}{\partial x^i} &= -\frac{(n-2)2^{(n-2)/2}}{|x|^n}x_i\left(a_0+\frac{a_jx_j}{|x|^2}\right)+\frac{a_i}{|x|^n}\\ &\quad-\frac{2x_i}{|x|^{n+2}}a_jx_j+O\left(\frac{1}{|x|^{n+1}}\right).\end{aligned}$$

Now we define w in terms of

$$(F^{-1})^*\tilde g = w^{4/(n-2)}g_{\mathbb{R}^n},$$

and set

$$a_0(t)=\tilde u(q_0,t),\quad a_i(t)=\frac{\partial(\tilde u(\cdot,t)\circ G)}{\partial x^i}(o),\quad a_{ij}(t)=\frac{\partial^2(\tilde u(\cdot,t)\circ G)}{\partial x^i\partial x^j}(o).$$

We call the point $y(t)$ with coordinates

$$y_i(t)=\frac{a_i(t)}{(n-2)a_0(t)}$$

the *center* of $w(\cdot,t)$. Note that w satisfies the flow equation (1.4) with g_0 replaced by $g_{\mathbb{R}^n}$, i.e.,

$$\tag{2.3}\frac{\partial w^N}{\partial t}=\Delta w+c(n)sw^N.$$

By Lemma 2, the expansion (2.2) holds for $w(\cdot,t)$ with $a_0=a_0(t)$, $a_i=a_i(t)$, and $a_{ij}=a_{ij}(t)$. It is important to notice that the expansion is uniform for all $t\in[0,T]$, where T is any given number in $[0,T^*)$.

Proposition 2. *There is a constant* $C>0$ *depending only on* $\operatorname{dist}(q_0,\Gamma)$, $\operatorname{diam}(\Gamma)$, $\operatorname{dist}(q_0,\partial\Phi(V))$ *and the constant in* (2.1) *such that*

$$|y(t)|\le C$$

for all $t\in[0,T^*)$.

Proof. Consider a given $T\in(0,T^*)$. Performing a rotation of coordinates and the transformation $x_n\mapsto -x_n$ if necessary, we may assume $y_n(T)=\max_i|y_i(T)|$. By the expansion (2.2) for $w(\cdot,t)$ and the arguments for Lemma 4.2 in [6] we derive that for some $\lambda_0\ge 1$ the following holds: for each $\lambda\ge\lambda_0$,

$$\tag{2.4} w_0(x)>w_0(x^\lambda)\quad\text{whenever } x_n<\lambda,$$

where $w_0 = w(\cdot, 0)$ and $x^\lambda = (x', 2\lambda - x_n)$ if we write $x = (x', x_n)$ (thus x^λ is the reflection of x about the plane $x_n = \lambda$). Note that although w_0 is only defined on $\mathbb{R}^n \backslash F(\Gamma)$, the arguments in [6] still apply here by virtue of Lemma 1. It is not hard to see that λ_0 can be estimated from above in terms of the constant C in (2.1), $\mathrm{dist}(q_0, \Gamma)$, $\mathrm{diam}(\Gamma)$, and $\mathrm{dist}(q_0, \partial\Phi(V))$.

We may assume that $F(\Gamma)$ lies strictly below the plane $x_n = \lambda_0$. Then we claim $y_n(T) \le \lambda_0$. By the same arguments as those for (2.4) and the fact that the expansion (2.2) for $w(\cdot, t)$ is uniform for all $t \in [0, T]$, there is some $\lambda_1 \ge \lambda_0$ such that for each $\lambda \ge \lambda_1$

$$w(x, t) > w(x^\lambda, t) \quad \text{whenever } t \in [0, T] \text{ and } x_n < \lambda. \tag{2.5}$$

Now we begin the procedure of moving the plane $x_n = \lambda$ by decreasing λ as in [6]. Set $w^\lambda(x, t) = w(x^\lambda, t)$. Then w^λ solves (2.3) and coincides with w along the plane $x_n = \lambda$. We restrict w^λ to the region $x_n \le \lambda$, $x \notin F(\Gamma)$, $0 \le t \le T$, and define

$$I = \{\lambda : \lambda > \lambda_0,\ \lambda > \max_{0 \le t \le T} y_n(t),\ w^\lambda \le w\}.$$

By (2.5), I is nonempty. I is also open. Indeed, $w^\lambda \equiv w$ can never happen for $\lambda \ge \lambda_0$ because of (2.4). (One may also use the singular set $F(\Gamma)$ to rule out $w^\lambda \equiv w$.) Hence for a given $\lambda \in I$, the maximum principle (the Harnack inequality for linear parabolic equations) implies

$$w^\lambda < w \quad \text{for } x_n < \lambda, \tag{2.6}$$

and the (parabolic version of the) Hopf boundary point lemma implies

$$\partial w / \partial x^n < 0 \quad \text{along the plane } x_n = \lambda. \tag{2.7}$$

For each fixed $t \in [0, T]$, we shift the origin to $y(t)$ to obtain the new expansion for $w(\cdot, t)$ in the new coordinates

$$\begin{aligned} w(\cdot, t) &= \frac{2^{(n-2)/2}}{|x|^{n-2}} \left(a_0 + (a_{ij} - \frac{n-2}{2} a_0 \delta_{ij}) \frac{x_i x_j}{|x|^4} + O\left(\frac{1}{|x|^3}\right) \right), \\ \frac{\partial w(\cdot, t)}{\partial x_i} &= -\frac{(n-2) 2^{(n-2)/2}}{|x|^n} a_0 x_i + O\left(\frac{1}{|x|^{n+1}}\right). \end{aligned} \tag{2.8}$$

The plane $x_n = \lambda$ becomes the plane $x_n = \lambda - y_n(t)$ in the new coordinates. Because $\lambda \in I$, we have $\lambda - y_n(t) > 0$. Hence we can argue as in [6] to show that there is an $\varepsilon(t) > 0$ with the following property: *If* $\lambda' \in (\lambda - \varepsilon(t), \lambda + \varepsilon(t))$, *then* $w^{\lambda'}(\cdot, t) \le w(\cdot, t)$.

Here, in addition to the arguments in [6], we also need to use Lemma 1. Since $\lambda > \max_{0\le t\le T} y_n(t)$ and the expansion (2.8) is uniform for all $t \in [0, T]$, we can choose $\varepsilon(t)$ uniformly for all $t \in [0, T]$. It follows that $(\lambda - \varepsilon, \lambda + \varepsilon) \subset I$ for some $\varepsilon > 0$. Thus the openness of I has been shown. Next we prove that I is closed in (λ_0, ∞).

Let $\lambda > \lambda_0$ be in the closure of I. By continuity, we have $w^\lambda \le w$ and $\lambda \ge \max_{0\le t\le T} y_n(t)$. If $\lambda = \max_{0\le t\le T} y_n(t)$, then $\lambda = y_n(t_0)$ for some $t_0 \in [0, T]$. Now we choose $y(t_0)$ as the new origin, denote $F(\Gamma)$ in the new coordinates by $\widetilde{\Gamma}$, and consider the stereographic projection $F : S^n \to \mathbb{R}^n$. Define z and z^λ in the following way:

$$F^*(w^{4/(n-2)} g_{_{\mathbb{R}^n}}) = z^{4/(n-2)} g_{_{S^n}}, \qquad F^*((w^\lambda)^{4/(n-2)} g_{_{\mathbb{R}^n}}) = (z^\lambda)^{4/(n-2)} g_{_{S^n}}.$$

Then z, z^λ are defined on $(S^n_+ \backslash F^{-1}(\widetilde{\Gamma})) \times [0, T]$, where S^n_+ is a hemisphere. The functions z and z^λ satisfy (1.4) with g_0 replaced by $g_{_{S^n}}$. We also know that $z^\lambda \le z$, and z^λ coincides with z along ∂S^n_+. Moreover, Lemma 2 and expansion (2.8) imply that

$$\frac{\partial z(\cdot, t_0)}{\partial \nu} \text{ (north pole)} \;=\; \frac{\partial z^\lambda(\cdot, t_0)}{\partial \nu} \text{ (north pole)} \;=\; 0,$$

where ν denotes the inward unit normal of ∂S^N_+. By the Hopf boundary point lemma we then deduce that $z \equiv z^\lambda$. This implies that $w \equiv w^\lambda$, which is impossible because of (2.4). Thus we conclude that $\lambda > \max\limits_{0\le t\le T} y_n(t)$, whence $\lambda \in I$. This shows that I is closed. We infer that $I = (\lambda_0, \infty)$. This proves our claim that $y_n(T) \le \lambda_0$. Hence $|\, y(T)\,| \le C$. Since T is arbitrary, the proposition is proven. q.e.d.

Now we can derive the Harnack inequality for the solution u. Proposition 2 immediately implies the estimate

$$|\,\nabla_{_{S^n}} \tilde{u}\,| / \tilde{u} \le C$$

on $\Phi(V')$ for some $C > 0$, where V' is a neighborhood of $\overline{p}_0$ with $V' \subset\subset V$, and S^n indicates the metric $g_{_{S^n}}$. Because of (2.1), this leads to

$$|\,\nabla_{g_0} u\,| / u \le C$$

on $\pi(V')$ for some $C > 0$. Since M is compact, we can cover M by finitely many such V' and thereupon obtain the above estimate on M for a larger C.

Next we consider the case that $(M, [g_0])$ is conformally covered by S^n. (Here it is not necessary to assume scalar positivity.) We lift g and

u to S^n. Then it is easy to see that the above arguments apply. In fact, they are simplified here because no singular set is present. Summarizing, we state

Theorem 4. *Assume that* $(M, [g_0])$ *is locally conformally flat and that* $[g_0]$ *is scalar positive. Choose a background metric* $g_0 \in [g_0]$. *If* g *is a solution of* (1.2) *with initial metric* $g^0 \in [g_0]$ *and* u *denotes the corresponding solution of* (1.4), *then*

$$\sup \frac{|\nabla_{g_0} u|}{u} \leq C, \tag{2.9}$$

where C *is a positive constant depending only on* g^0, g_0, *and the conformal properties of* $(M, [g_0])$. *For each* t, *integrating* (2.9) *along a shortest geodesic between a maximum point and a minimum point of* $u(\cdot, t)$ *yields*

$$\inf_t u \geq c \sup_t u \tag{2.10}$$

for some $c > 0$.

This result immediately implies Theorem 1.

Proof of Theorem 1. We start with the solution g, u given before on the maximal time interval $[0, T^*)$. We compute the change rate of volume

$$\frac{dV}{dt} = \int_M \frac{\partial}{\partial t}(dv) = \frac{1}{2}\int_M \operatorname{trace}\left(\frac{\partial g}{\partial t}\right) dv = 0.$$

Hence the volume stays constant. But

$$V(t) = \int_M u(\cdot, t)^{2n/(n-2)} dv_{g_0}.$$

Thus the Harnack inequality (2.10) implies that u is uniformly bounded from above and away from zero. By results of Krylov and Safonov [9] this leads to a Hölder continuity estimate for u on $M \times [\min(1, T^*/2), T^*)$. Standard linear theory and bootstrapping then yield smooth estimates for u on $M \times [\min(1, T^*/2), T^*)$. It follows that $T^* = \infty$, since otherwise we would be able to extend u beyond T^* by Proposition 1. By Simon's general results [18], u converges smoothly to a unique limit u_∞ as $t \to \infty$. Consequently, g converges smoothly to a unique limit g_∞ as $t \to \infty$. On the other hand, the formula for the gradient of the total scalar curvature functional $\mathbf{S}$ implies

$$\frac{ds}{dt} = -\frac{n-2}{2V}\int_M (R - s)^2 dv. \tag{2.11}$$

It follows that

$$\int_0^\infty \int_M (R-s)^2 \, dv \, dt < \infty. \tag{2.12}$$

We easily see that the limit metric g_∞ has constant scalar curvature.

3. The scalar negative and flat cases

We first deal with the scalar negative case. We choose the background metric g_0 such that $R_{g_0} < 0$. Let $g^0 \in [g_0]$ be an initial metric and g the solution of (1.2) with $g(0) = g^0$ on a maximal time interval $[0, T^*)$. First apply the maximum principle to (1.4) to derive

$$\frac{du^N_{\min}(t)}{dt} \ge c(n) \min |R_{g_0}| \, u_{\min}(t) + c(n) s u^N_{\min}(t), \tag{3.1}$$

where $u_{\min}(t) = \min_t u$. We have $s \ge \alpha V^{-2/n}$, where α denotes the infimum of the total scalar curvature $\mathbf{S}$ on $[g_0]$. The conformal invariant α is finite by formula (1.6) and the Hölder inequality. Hence we obtain, by integrating (3.1),

$$u^{N-1}_{\min}(t) \ge \min\{u_{\min}(0)^{N-1}, \ |\alpha|^{-1} \min |R_{g_0}| \, V^{2/n}\}. \tag{3.2}$$

On the other hand, the maximum principle also implies

$$\frac{du^N_{\max}(t)}{dt} \le -c(n)(\min R_{g_0}) u_{\max}(t) + c(n) s u^N_{\max}(t), \tag{3.3}$$

where $u_{\max}(t) = \max_t u$. By (2.11), $s \le s(0)$. Consequently,

$$u^N_{\max}(t) \le (u^N_{\max}(0) + 1) e^{c(n)(|\min R_{g_0}| + s(0))t}. \tag{3.4}$$

Note that this estimate holds without the scalar negativity assumption. The estimates (3.2) and (3.4) imply $T^* = \infty$. Indeed, if $T^* < \infty$, then by (3.2) and (3.4) u would be uniformly bounded from above and away from zero. Then u would extend beyond T^*.

Next we claim that s will eventually become negative, even if it may not be so at the start. In fact, if s remains nonnegative always, then (3.1) would imply

$$\frac{du^N_{\min}(t)}{dt} \ge c(n) \min |R_{g_0}| \, u_{\min}(t),$$

whence $u_{\min}(t)$ approaches infinity as $t \to \infty$. This contradicts the constancy of volume. Choosing a later time as the time origin, we may assume $s(0) < 0$. Then $s \leq -|s(0)|$ by (2.11). Hence (3.3) yields

$$u_{\max}^{N-1}(t) \leq \max\{u_{\max}(0)^{N-1}, \frac{1}{|s(0)|} \max |R_{g_0}|\}, \tag{3.5}$$

which together with (3.2) implies that u is uniformly bounded from above and away from zero. Therefore we obtain uniform smooth estimates on u. By (2.12) we can get a limit g_∞ of g such that g_∞ has constant negative scalar curvature. Then for some time T, $g(T)$ has negative scalar curvature. From this moment we can argue as in [8], and observe that the scalar curvature R satisfies the evolution equation

$$\frac{\partial R}{\partial t} = (n-1)\Delta R + R(R-s), \tag{3.6}$$

which follows from a computation similar to (but simpler than) that in [7], (or alternatively, also from the formulas for curvature deformations in [1].) The maximum principle argument in [8] then shows that R approaches s exponentially. From (1.2), it follows that u converges exponentially in the C^0 norm as $t \to \infty$. Then u and, hence, g converge smoothly at an exponential rate.

Next we treat the scalar flat case. We use the above notation, but this time we can assume $R_{g_0} \equiv 0$. Note that s can never be negative. Otherwise, formula (1.6) would imply that the first eigenvalue of L_{g_0} is negative. If φ denotes a positive first eigenfunction, $\varphi^{4/(n-2)} g_0$ would then have negative scalar curvature, which cannot happen because of the scalar flat assumption. If s is zero at the start, it remains so. Formula (2.11) then implies that R has to be identically zero for all time. Thus the solution of Yamabe flow is constant in time. Next we assume that $s(0) > 0$. We observe

$$\frac{u_{\min}^N(t)}{u_{\min}^N(0)} \geq c(n) \int_0^t s\, dt$$

and

$$\frac{u_{\max}^N(t)}{u_{\max}^N(0)} \leq c(n) \int_0^t s\, dt,$$

which are consequences of (3.1) and (3.3). Hence we obtain the Harnack inequality

$$u_{\min}(t) \geq \frac{u_{\min}(0)}{u_{\max}(0)} u_{\max}(t).$$

It follows that u exists for all time, and uniform smooth estimates on u hold. Another consequence is that $s \to 0$ as $t \to \infty$. To produce exponential convergence, we look at integral quantities as in [8]. Multiplying (1.4) by $u^{1-N}\Delta_{g_0} u$ and then integrating the resulting equation, we deduce

$$N\frac{d}{dt}\int_M |\nabla_{g_0} u|^2 \, dv_{g_0} + 2\int_M |\Delta_{g_0} u|^2 u^{1-N} \, dv_{g_0} = 2c(n)s\int_M |\nabla_{g_0} u|^2 \, dv_{g_0}. \tag{3.7}$$

Using the inequality

$$\int_M |\Delta_{g_0} u|^2 \, dv_{g_0} \geq c\int_M |\nabla_{g_0} u|^2 \, dv_{g_0}$$

for some $c > 0$ and the fact $\lim s = 0$ we infer

$$\int_M |\nabla_{g_0} u|^2 \, dv_{g_0} \leq Ce^{-ct}. \tag{3.8}$$

(C and c always denote some positive constants.) Now by integrating (3.7) we arrive at

$$\int_T^\infty \int_M |\Delta_{g_0} u|^2 \, dv_{g_0} \leq Ce^{-cT}$$

for each $T > 0$, which implies

$$\int_T^\infty \int_M R^2 \, dv \leq Ce^{-cT}.$$

Hence at some point in each interval $T \leq t \leq T+1$ we have

$$\int_M R^2 \, dv \leq Ce^{-ct}.$$

Since s decreases, this yields

$$s \leq Ce^{-ct}$$

for all t. Integrating (1.4) we then see that the integral $\int_M u^N \, dv_{g_0}$ converges exponentially. On the other hand, estimate (3.8) and the Poincaré inequality imply that u^N converges to its average exponentially in the L^2 norm. Thus u^N converges to its average smoothly at an exponential rate. It follows that g converges exponentially, and the proof of Theorem 2 is finished.

4. Long time existence

We first present two lemmas.

Lemma 3. *Let $C > 0$ and $\varepsilon > 0$ be given. Let u be a positive smooth solution of* (1.4) *on some time interval* $[0, T]$ *with $u \le C$, $|s| \le C$, and $T > \varepsilon$. Then the modulus of continuity of u on $M \times [\varepsilon, T]$ can be estimated in terms of C, ε, n, and the background metric g_0.*

Proof. This follows from [12]; see also [4].

Lemma 4. *Let u be a positive solution of* (1.4) *on some time interval* $[0, T)$ *with $T < \infty$. Then u extends continuously to T and the extension is positive everywhere.*

Proof. The continuous extension of u is guaranteed by Lemma 3, inequality (3.4), and the estimate $\alpha V^{-2/n} \le s \le s(0)$. Since the volume remains constant and is nonzero, the extension of u at T cannot vanish identically. Assume that $u(\cdot, T)$ is positive at a point p_0, we are going to show that $u(\cdot, T)$ is positive in a uniform neighborhood of p_0. By the connectedness of M, this uniform positivity then implies that u is positive everywhere at time T. Our argument is inspired by [10] and [11].

We work with the Riemannian manifold (M, g_0). Let $\mathbf{e}$ be a unit tangent vector at p_0. Consider the point $q_0 = \exp_{p_0}(r_0\mathbf{e})$, where r_0 is half the injectivity radius of (M, g_0). There is some $\delta > 0$ such that $u(\cdot, T)$ is positive on the spherical region $D = B_\delta(p_0) \cap \partial B_{r_0}(q_0)$, where $B_r(p)$ denotes the closed geodesic ball with radius r and center p. Let r denote the distance to q_0. Then the Laplacian of g_0 can be written

$$\Delta_{g_0} = \frac{\partial^2}{\partial r^2} + H(r)\frac{\partial}{\partial r} + \Delta_{S_r}, \tag{4.1}$$

where $H(r)$ is the mean curvature function of $S_r = \partial B_r(q_0)$, and Δ_{S_r} is the Laplacian on S_r for the induced metric. Now consider the geodesic cone Q with tip q_0 and cap D, and set $D_r = Q \cap S_r$. Let $\varphi(r, \cdot)$ denote the positive first eigenfunction of $r^2\Delta_{S_r}$ on D_r with zero boundary values and integral one. We put $v = r^b\varphi$, where $b > 0$ is to be determined. We calculate

$$\begin{aligned}\Delta_{g_0} v = b(b-1)r^{b-2}\varphi + 2br^{b-1}\frac{\partial\varphi}{\partial r} + r^b\frac{\partial^2\varphi}{\partial r^2} \\ + H(r)(br^{b-1}\varphi + r^b\frac{\partial\varphi}{\partial r}) - \lambda(r)r^{b-2}\varphi,\end{aligned} \tag{4.2}$$

where $\lambda(r)$ is the first eigenvalue of $r^2\Delta_{S_r}$ on D_r. It is easy to see that $r^2\Delta_{S_r}$ converges to the Laplacian on the unit Euclidean sphere as $r \to 0$. In particular, $\lambda(r)$ is uniformly bounded with respect to r. By the Hopf boundary point lemma,

$$\partial\varphi/\partial\nu > 0,$$

where ν denotes the inward unit normal of D_r. With a little care one actually obtains

$$\partial\varphi/\partial\nu > \varepsilon, \tag{4.3}$$

where $\varepsilon > 0$ depends only on δ and the local geometry near q_0. On the other hand, it is not hard to derive a priori estimates on $\partial\varphi/\partial r$ and $\partial^2\varphi/\partial r^2$ which depend on δ and the local geometry near q_0. Since $\partial\varphi/\partial r$ and $\partial^2\varphi/\partial r^2$ have zero boundary values, the positivity of φ and estimate (4.3) imply

$$\left|\frac{\partial\varphi}{\partial r}\right| \le C\varphi, \qquad \left|\frac{\partial^2\varphi}{\partial r^2}\right| \le C\varphi$$

for some constant $C > 0$. We also note that φ is uniformly bounded with respect to r and that $|H(r)| \le C'/r$ for some $C' > 0$ depending on the geometry of (M, g_0). With all the above information, it is straightforward to see that

$$Lv \ge 0,$$

whenever $r \le r_0$, and b is chosen sufficiently large. Now we fix some $T' < T$ and choose a small $\varepsilon' > 0$ such that $\varepsilon' v \le u$ on $D \times [T', T]$ and $Q \times \{T'\}$. The inequality $\varepsilon' v \le u$ also holds on the remaining part of the parabolic boundary of $Q \times [T', T]$, because v vanishes there. Hence the maximum principle implies that $u \ge \varepsilon' v$ on $Q \times [T, T]$; consequently u is positive along the geodesic $\exp_{p_0}(t\mathbf{e})$, $0 \le t < r_0$. Since $\mathbf{e}$ is arbitrary, we conclude that u is positive in $B_{r_0}(p_0)$. q.e.d.

With these two lemmas, the long time existence of Yamabe flow follows easily.

Proof of Theorem 3. Let u be a positive smooth solution of (1.4) on a maximal time interval $[0, T^*)$. If $T^* < \infty$, then u extends to a continuous positive function on $M \times [0, T^*]$. It follows that u is uniformly bounded from above and away from zero. Hence u extends smoothly beyond T^*, contradicting the maximality of T^*.

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